On the Excess of the Number of Combinations in a Set which have an Even Number of Inversions over those which have an Odd Number.

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1. If we are given any combination of n numbers m at a time, the combination of the remaining $n - m$ numbers is said to be the complementary with respect to n of the given combination.

Let it be understood (unless otherwise expressed) that the numbers in any combination are arranged in their natural order (order of magnitude). Let $(n \,|\, m)_1$, $(n \,|\, m)_2 \ldots . (n \,|\, m)_\mu$ denote the $\dfrac{n(n-1) \ldots . (n-m+1)}{m\,!} = n_m = \mu$ combinations of the numbers $1, 2, 3 \ldots n$ taken m at a time, and let $(\bar{n} \,|\, m)_1 \ldots (\bar{n} \,|\, m)_\mu$ denote their complementaries. Let $(n \,|\, m \,|\, l)_1$, $(n \,|\, m \,|\, l)_2 \ldots . (n \,|\, m \,|\, l)_\lambda$ denote the $n_l = \lambda$ combinations of the numbers in the combination $(n \,|\, m)_a$ taken l at a time, and let $(n \,|\, \bar{m} \,|\, l)_{a\,\beta}$ denote the combination which is the complementary with respect to m of the combination $(n \,|\, m \,|\, l)_{a\,\beta}$, i. e., the combination of the $m - l$ numbers remaining after the numbers in the combination $(n \,|\, m \,|\, l)_{a\,\beta}$ are taken out of the combination $(n \,|\, m)_a$. For present purposes let $(n \,|\, \bar{m} \,|\, l)_{a\,\beta}(n \,|\, m \,|\, l)_{a\,\beta}$ denote the combination made up of the numbers in $(n \,|\, \bar{m} \,|\, l)_{a\,\beta}$ followed by the numbers in $(n \,|\, m \,|\, l)_{a\,\beta}$. In contrast with this I have used elsewhere * $(n \,|\, \bar{m} \,|\, l)_{a\,\beta}(n \,|\, m \,|\, l)_{a\,\beta}$ to denote the com-

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bination of the numbers in the two combinations $(n\,\overline{m}\,|\,l)$ and $(n\,|\,m\,|\,l)$ arranged in their natural order.

Let any combination having an odd number of inversions from the natural order be affected with the negative sign.

2. If k denote the number of inversions in $(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l)$, then it may be easily proven that the number of inversions in $(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l)$ is $l\,(m-l)-k$.

Therefore

$$(n\,|\,m\,|\,l)(n\,|\,\overline{m}\,|\,l) = (-1)^{l(m-l)-2k}\,(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l)$$

$$= (-1)^{l(m-l)}\,(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l).$$

3. The combination $(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l) = \pm\,(n\,|\,m)$ according as the number of inversions is even or odd.

Let

$$(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l) + (n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l) + \ldots + (n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l)$$

$$= \phi\,(m,\,l)\,.\,(n\,|\,m), \qquad (1)$$

then will

$$(n\,|\,m\,|\,l)(n\,|\,\overline{m}\,|\,l) + (n\,|\,m\,|\,l)(n\,|\,\overline{m}\,|\,l) + \ldots + (n\,|\,m\,|\,l)(n\,|\,\overline{m}\,|\,l) = \phi\,(m,\,m-l)\,.\,(n\,|\,m).$$

But

$$(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l) = (-1)^{l(m-l)}\,(n\,|\,m\,|\,l)(n\,|\,\overline{m}\,|\,l), \qquad \text{(art. 2)}$$

$$\therefore\ \phi\,(m,\,l) = (-1)^{l(m-l)}\,\phi\,(m,\,m-l).$$

If $l = 1$, the signs of the left-hand member of equation (1) are evidently alternately positive and negative, therefore,

$$\phi\,(m,\,1) = 1 \text{ or } 0$$

according as m is odd or even.

It is also apparent that

$$\phi\,(m,\,m) = 1.$$

4. The set of combinations

$$(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l),\quad (n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l),\ \ldots\ ,(n\,|\,\overline{m}\,|\,l)(n\,|\,m\,|\,l)$$

may be divided up into groups as follows:

The first group containing the first $(m-1)_{l-1}$ combinations,
" second " " " next $(m-2)_{l-1}$ "
....
" $(m-l+1)^{\text{st}}$ " " " last $(l-1)_{l-1}=1$ "

The first number in the second part of each combination of the rth group is the same, and is the rth of the selection of m numbers, i. e., the rth of the numbers in $(n|m)$. The first $r-1$ numbers in the first part of each combina-
tion of the rth group are the same and are the first $r-1$ of the numbers in $(n|m)$. It follows from this that the signs of the combinations of the rth group
are the same as or the opposite to (according as $m-l-r+1$ is even or odd, there being $m-l-r+1$ numbers in the first part greater than the first number in the second part) the signs of the corresponding members of the set obtained by striking out the $r-1$ numbers common to the first part and the one number common to the second part of each combination of the group.

We have, therefore

$$\varphi(m, l) = \varphi(l-1, l-1) - \varphi(l, l-1)\,\varphi(l+1, l-1)\ldots.$$
$$+ (-1)^{m-l-r+1}\varphi(m-r, l-1) + \ldots. + (-1)^{m-l}\varphi(m-1, l-1), \quad (2)$$

a reduction formula for $\varphi(m, l)$.

As an immediate consequence of equation (2), we have

$$\varphi(m, l) = \varphi(m-r, l) + (-1)^{m-r-l+1}\varphi(m-r, l-1)$$
$$+ (-1)^{m-l}\varphi(m-1, l-1)$$
$$= \varphi(m-1, l) + (-1)^{m-l}\varphi(m-1, l-1). \quad (3)$$

5. By successive applications of equation (2), we have

$$\varphi(2m, 2l+1) = \varphi(2l, 2l) - \varphi(2l+1, 2l) + \ldots. - \varphi(2m-1l\,2l)$$
$$= \varphi(2l-1, 2l-1)$$
$$- \varphi(2l-1, 2l-1) + \varphi(2l, 2l-1)$$
$$+ \varphi(2l-1, 2l-1) - \varphi(2l, 2l-1) + \varphi(2l+1, 2l-l)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$- \varphi(2l-1, 2l-1) + \ldots. \qquad + \varphi(2m-2, 2l-1)$$
$$= \varphi(2l, 2l-1) + \varphi(2l+2, 2l-1) + \ldots.$$
$$+ \varphi(2m-2, 2l-1). \quad (4)$$

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Put $l = 1$; then

$$\phi(2m, 3) = \phi(2, 1) + \phi(4, 1) + \ldots + \phi(2m - 2, 1)$$
$$= 0. \qquad \text{(art. 3)}.$$

Put $l = 2$, then

$$\phi(2m, 5) = \phi(4, 3) + \phi(6, 3) + \ldots + \phi(2m - 2, 3)$$
$$= 0.$$

In this way it may be shown that

$$\phi(2m, 2l + 1) = 0, \qquad (l = 1, 2 \ldots \overline{m - 1}). \quad (5)$$

6. From equations (3) and (5), we have

$$\phi(2m + 1, 2l) = \phi(2m, 2l) - \phi(m, 2l - 1)$$
$$= \phi(2m, 2l),$$
$$\phi(2m + 1, 2l + 1) = \phi(2m, 2l + 1) + \phi(2m, 2l)$$
$$= \phi(2m, 2l).$$
$$\phi(2m + 1, 2l + 1) = \phi(2m + 1, 2l) = \phi(2m, 2l). \quad (6)$$

It follows from this and art. 3 that

$$\phi(m, l) = \phi(m, m - l).$$

If $l = m$, then

$$\phi(m, m) = \phi(m, 0) = 1.$$

7. From equations (2), (5) and (6), we have

$$\phi(2m, 2l) = \phi(2m - 1, 2l - 1) + \phi(2m - 3, 2l - 1) + \ldots$$
$$+ \phi(2l - 1, 2l - 1)$$
$$= \phi(2m - 1, 2l - 2) + \phi(2m - 4, 2l - 2) + \ldots$$
$$+ \phi(2l - 2, 2l - 2). \quad (7)$$

These properties at once suggest that

$$\phi(2m, 2l) = m_l,$$

and it may be easily proved that this is true.

If, in equation (7), we put—

1st. $l = 1$, then

$$\phi(2m, 2) = \phi(2m - 2, 0) + \phi(2m - 4, 0) + \ldots + \phi(0, 0)$$
$$= m \text{ or } m_1;$$

2nd. $l = 2$, then

$$\phi(2m, 4) = \phi(2m - 2, 2) + \phi(2m - 4, 2) + \ldots + \phi(2, 2)$$
$$= (m - 1)_1 + (m - 2)_1 \qquad + \ldots + 1_1$$
$$= m_2 ;$$

3rd. $l = 3$, then

$$\phi(2m, 6) = \phi(2m - 2, 4) + \phi(2m - 4, 4) + \ldots + \phi(4, 4)$$
$$= (m - 1)_2 + (m - 2)_2 \qquad + \ldots + 2_2$$
$$= m_3 .$$

In this way we see from equation (7) itself, that if it is true for any value of l, it is true for a value one greater and, therefore, true for all values. Hence

$$\phi(2m, 2l) = m_l, \qquad\qquad (l = 1, 2 \ldots m)$$